

On the Second Approximate Matsumoto Metric

A. Tayebi, T. Tabatabaeifar and E. Peyghan

August 4, 2014

Abstract

In this paper, we study the second approximate Matsumoto metric $F = \alpha + \beta + \beta^2/\alpha + \beta^3/\alpha^2$ on a manifold M . We prove that F is of scalar flag curvature and isotropic S-curvature if and only if it is isotropic Berwald metric with almost isotropic flag curvature.

Keywords: Isotropic Berwald curvature, S-curvature, almost isotropic flag curvature.¹

1 Introduction

The flag curvature in Finsler geometry is a natural extension of the sectional curvature in Riemannian geometry, which is first introduced by L. Berwald. For a Finsler manifold (M, F) , the flag curvature is a function $\mathbf{K}(P, y)$ of tangent planes $P \subset T_x M$ and directions $y \in P$. F is said to be of scalar flag curvature if the flag curvature $\mathbf{K}(P, y) = \mathbf{K}(x, y)$ is independent of flags P associated with any fixed flagpole y . F is called of almost isotropic flag curvature if

$$\mathbf{K} = \frac{3c_x y^m}{F} + \sigma, \quad (1)$$

where $c = c(x)$ and $\sigma = \sigma(x)$ are scalar functions on M . One of the important problems in Finsler geometry is to characterize Finsler manifolds of almost isotropic flag curvature [11].

To study the geometric properties of a Finsler metric, one also considers non-Riemannian quantities. In Finsler geometry, there are several important non-Riemannian quantities: the Cartan torsion \mathbf{C} , the Berwald curvature \mathbf{B} , the mean Landsberg curvature \mathbf{J} and S-curvature \mathbf{S} , etc [6][9][11][18]. These are geometric quantities which vanish for Riemannian metrics.

Among the non-Riemannian quantities, the S-curvature $\mathbf{S} = \mathbf{S}(x, y)$ is closely related to the flag curvature which constructed by Shen for given comparison theorems on Finsler manifolds. A Finsler metric F is called of isotropic S-curvature if

$$\mathbf{S} = (n+1)cF, \quad (2)$$

¹ 2010 Mathematics subject Classification: 53C60, 53C25.

for some scalar function $c = c(x)$ on M . In [11], it is proved that if a Finsler metric F of scalar flag curvature is of isotropic S-curvature (2), then it has almost isotropic flag curvature (1).

The geodesic curves of a Finsler metric $F = F(x, y)$ on a smooth manifold M , are determined by $\ddot{c}^i + 2G^i(\dot{c}) = 0$, where the local functions $G^i = G^i(x, y)$ are called the spray coefficients. A Finsler metric F is called a Berwald metric, if G^i are quadratic in $y \in T_x M$ for any $x \in M$. A Finsler metric F is said to be isotropic Berwald metric if its Berwald curvature is in the following form

$$B^i_{jkl} = c \left\{ F_{y^j y^k} \delta^i_l + F_{y^k y^l} \delta^i_j + F_{y^l y^j} \delta^i_k + F_{y^j y^k y^l} y^i \right\}, \quad (3)$$

where $c = c(x)$ is a scalar function on M [6].

As a generalization of Berwald curvature, Bácsó-Matsumoto proposed the notion of Douglas curvature [1]. A Finsler metric is called a Douglas metric if $G^i = \frac{1}{2} \Gamma^i_{jk}(x) y^j y^k + P(x, y) y^i$.

In order to find explicit examples of Douglas metrics, we consider (α, β) -metrics. An (α, β) -metric is a Finsler metric of the form $F := \alpha \phi(\frac{\beta}{\alpha})$, where $\phi = \phi(s)$ is a C^∞ on $(-b_0, b_0)$ with certain regularity, $\alpha = \sqrt{a_{ij}(x) y^i y^j}$ is a Riemannian metric and $\beta = b_i(x) y^i$ is a 1-form on M . This class of metrics is were first introduced by Matsumoto [10]. Among the (α, β) -metrics, the Matsumoto metric is special and significant metric which constitute a majority of actual research. The Matsumoto metric is expressed as

$$F = \alpha \left[1 + \frac{\beta}{\alpha} + \left(\frac{\beta}{\alpha} \right)^2 + \left(\frac{\beta}{\alpha} \right)^3 + \cdots \right].$$

This metric was introduced by Matsumoto as a realization of Finsler's idea "a slope measure of a mountain with respect to a time measure" [19]. In the Matsumoto metric, the 1-form $\beta = b_i y^i$ was originally to be induced by earth gravity. Hence, we could regard $b_i(x)$ as the the infinitesimals and neglect the infinitesimals of degree of $b_i(x)$ more than two [12][13][14][15][16]. An approximate Matsumoto metric is a Finsler metric in the following form

$$F = \alpha \left[\sum_{k=0}^r \left(\frac{\beta}{\alpha} \right)^k \right], \quad (4)$$

where $|\beta| < |\alpha|$ (for more information, see [13]). This metric was introduced by Park-Choi in [13]. By definition, the Matsumoto metric is expressed as $\lim_{r \rightarrow \infty} L(\alpha, \beta) = \frac{\alpha^2}{\alpha - \beta}$.

In this paper, we consider second approximate Matsumoto metric $F = \alpha + \beta + \frac{\beta^2}{\alpha} + \frac{\beta^3}{\alpha^2}$ with some non-Riemannian curvature properties and prove the following.

Theorem 1.1. *Let $F = \alpha + \beta + \frac{\beta^2}{\alpha} + \frac{\beta^3}{\alpha^2}$ be a non-Riemannian second approximate Matsumoto metric on a manifold M of dimension n . Then F is of scalar flag curvature with isotropic S-curvature (2), if and only if it has isotropic Berwald curvature (3) with almost isotropic flag curvature (1). In this case, F must be locally Minkowskian.*

2 Preliminaries

Let M be a n -dimensional C^∞ manifold. Denote by $T_x M$ the tangent space at $x \in M$, by $TM = \cup_{x \in M} T_x M$ the tangent bundle of M , and by $TM_0 = TM \setminus \{0\}$ the slit tangent bundle on M . A Finsler metric on M is a function $F : TM \rightarrow [0, \infty)$ which has the following properties:

- (i) F is C^∞ on TM_0 ;
- (ii) F is positively 1-homogeneous on the fibers of tangent bundle TM ;
- (iii) for each $y \in T_x M$, the following quadratic form \mathbf{g}_y on $T_x M$ is positive definite,

$$\mathbf{g}_y(u, v) := \frac{1}{2} \frac{\partial^2}{\partial s \partial t} [F^2(y + su + tv)]|_{s,t=0}, \quad u, v \in T_x M.$$

Let $x \in M$ and $F_x := F|_{T_x M}$. To measure the non-Euclidean feature of F_x , define $\mathbf{C}_y : T_x M \otimes T_x M \otimes T_x M \rightarrow \mathbb{R}$ by

$$\mathbf{C}_y(u, v, w) := \frac{1}{2} \frac{d}{dt} [\mathbf{g}_{y+tw}(u, v)]|_{t=0}, \quad u, v, w \in T_x M.$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y \in TM_0}$ is called the Cartan torsion. It is well known that $\mathbf{C} = 0$ if and only if F is Riemannian [17]. For $y \in T_x M_0$, define mean Cartan torsion \mathbf{I}_y by $\mathbf{I}_y(u) := I_i(y)u^i$, where $I_i := g^{jk}C_{ijk}$. By Diecke Theorem, F is Riemannian if and only if $\mathbf{I}_y = 0$.

The horizontal covariant derivatives of \mathbf{I} along geodesics give rise to the mean Landsberg curvature $\mathbf{J}_y(u) := J_i(y)u^i$, where $J_i := I_i|_s y^s$. A Finsler metric is said to be weakly Landsbergian if $\mathbf{J} = 0$.

Given a Finsler manifold (M, F) , then a global vector field \mathbf{G} is induced by F on TM_0 , which in a standard coordinate (x^i, y^i) for TM_0 is given by $\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i}$, where

$$G^i := \frac{1}{4} g^{il} \left[\frac{\partial^2 (F^2)}{\partial x^k \partial y^l} y^k - \frac{\partial (F^2)}{\partial x^l} \right], \quad y \in T_x M.$$

The \mathbf{G} is called the spray associated to (M, F) . In local coordinates, a curve $c(t)$ is a geodesic if and only if its coordinates $(c^i(t))$ satisfy $\ddot{c}^i + 2G^i(\dot{c}) = 0$.

For a tangent vector $y \in T_x M_0$, define $\mathbf{B}_y : T_x M \otimes T_x M \otimes T_x M \rightarrow T_x M$ and $\mathbf{E}_y : T_x M \otimes T_x M \rightarrow \mathbb{R}$ by $\mathbf{B}_y(u, v, w) := B^i_{jkl}(y)u^j v^k w^l \frac{\partial}{\partial x^i}|_x$ and $\mathbf{E}_y(u, v) := E_{jk}(y)u^j v^k$ where

$$B^i_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}, \quad E_{jk} := \frac{1}{2} B^m_{jkm}.$$

The \mathbf{B} and \mathbf{E} are called the Berwald curvature and mean Berwald curvature, respectively. Then F is called a Berwald metric and weakly Berwald metric if $\mathbf{B} = \mathbf{0}$ and $\mathbf{E} = \mathbf{0}$, respectively.

A Finsler metric F is said to be isotropic mean Berwald metric if its mean Berwald curvature is in the following form

$$E_{ij} = \frac{n+1}{2F} ch_{ij}, \quad (5)$$

where $c = c(x)$ is a scalar function on M and h_{ij} is the angular metric [6].

Define $\mathbf{D}_y : T_x M \otimes T_x M \otimes T_x M \rightarrow T_x M$ by $\mathbf{D}_y(u, v, w) := D^i_{jkl}(y) u^i v^j w^k \frac{\partial}{\partial x^i} |_x$ where

$$D^i_{jkl} := B^i_{jkl} - \frac{2}{n+1} \{E_{jk} \delta^i_l + E_{jl} \delta^i_k + E_{kl} \delta^i_j + E_{jk, l} y^i\}.$$

We call $\mathbf{D} := \{\mathbf{D}_y\}_{y \in TM_0}$ the Douglas curvature. A Finsler metric with $\mathbf{D} = 0$ is called a Douglas metric. The notion of Douglas metrics was proposed by Bácsó-Matsumoto as a generalization of Berwald metrics [1].

For a Finsler metric F on an n -dimensional manifold M , the Busemann-Hausdorff volume form $dV_F = \sigma_F(x) dx^1 \cdots dx^n$ is defined by

$$\sigma_F(x) := \frac{\text{Vol}(\mathbb{B}^n(1))}{\text{Vol}\left\{(y^i) \in \mathbb{R}^n \mid F\left(y^i \frac{\partial}{\partial x^i} |_x\right) < 1\right\}}.$$

In general, the local scalar function $\sigma_F(x)$ can not be expressed in terms of elementary functions, even F is locally expressed by elementary functions. Let G^i denote the geodesic coefficients of F in the same local coordinate system. The S-curvature can be defined by

$$\mathbf{S}(\mathbf{y}) := \frac{\partial G^i}{\partial y^i}(x, y) - y^i \frac{\partial}{\partial x^i} \left[\ln \sigma_F(x) \right],$$

where $\mathbf{y} = y^i \frac{\partial}{\partial x^i} |_x \in T_x M$. It is proved that $\mathbf{S} = 0$ if F is a Berwald metric. There are many non-Berwald metrics satisfying $\mathbf{S} = 0$. \mathbf{S} said to be *isotropic* if there is a scalar functions $c(x)$ on M such that $\mathbf{S} = (n+1)c(x)F$.

The Riemann curvature $\mathbf{R}_y = R^i_k dx^k \otimes \frac{\partial}{\partial x^i} |_x : T_x M \rightarrow T_x M$ is a family of linear maps on tangent spaces, defined by

$$R^i_k = 2 \frac{\partial G^i}{\partial x^k} - y^j \frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j} \frac{\partial G^j}{\partial y^k}.$$

For a flag $P = \text{span}\{y, u\} \subset T_x M$ with flagpole y , the flag curvature $\mathbf{K} = \mathbf{K}(P, y)$ is defined by

$$\mathbf{K}(P, y) := \frac{\mathbf{g}_y(u, \mathbf{R}_y(u))}{\mathbf{g}_y(y, y) \mathbf{g}_y(u, u) - \mathbf{g}_y(y, u)^2}.$$

We say that a Finsler metric F is of scalar curvature if for any $y \in T_x M$, the flag curvature $\mathbf{K} = \mathbf{K}(x, y)$ is a scalar function on the slit tangent bundle TM_0 . In this case, for some scalar function \mathbf{K} on TM_0 the Riemann curvature is in the following form

$$R^i_k = \mathbf{K} F^2 \{\delta^i_k - F^{-1} F_{y^k} y^i\}.$$

If $\mathbf{K} = \text{constant}$, then F is said to be of constant flag curvature. A Finsler metric F is called *isotropic flag curvature*, if $\mathbf{K} = \mathbf{K}(x)$.

3 Proof of Theorem 1.1

Let $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$ be an (α, β) -metric, where $\phi = \phi(s)$ is a C^∞ on $(-b_0, b_0)$ with certain regularity, $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on a manifold M . Let

$$r_{ij} := \frac{1}{2} [b_{i|j} + b_{j|i}], \quad s_{ij} := \frac{1}{2} [b_{i|j} - b_{j|i}].$$

$$r_j := b^i r_{ij}, \quad s_j := b^i s_{ij}.$$

where $b_{i|j}$ denote the coefficients of the covariant derivative of β with respect to α . Let

$$r_{i0} := r_{ij}y^j, \quad s_{i0} := s_{ij}y^j, \quad r_0 := r_j y^j, \quad s_0 := s_j y^j.$$

Put

$$\begin{aligned} Q &= \frac{\phi'}{\phi - s\phi'}, \\ \Theta &= \frac{\phi\phi' - s(\phi\phi'' + \phi'^2)}{2\phi[(\phi - s\phi') + (b^2 - s^2)\phi'']} \\ \Psi &= \frac{\phi''}{2[(\phi - s\phi') + (b^2 - s^2)\phi'']}. \end{aligned} \tag{6}$$

Then the S -curvature is given by

$$\begin{aligned} \mathbf{S} &= [Q' - 2\Psi Qs - 2(\Psi Q)'(b^2 - s^2) - 2(n+1)Q\Theta + 2\lambda]s_0 \\ &\quad + 2(\Psi + \lambda)s_0 + \alpha^{-1}[(b^2 - s^2)\Psi' + (n+1)\Theta]r_{00}. \end{aligned} \tag{7}$$

Let us put

$$\begin{aligned} \Delta &:= 1 + sQ + (b^2 - s^2)Q', \\ \Phi &:= -(n\Delta + 1 + sQ)(Q - sQ') - (b^2 - s^2)(1 + sQ)Q''. \end{aligned}$$

In [5], Cheng-Shen characterize (α, β) -metrics with isotropic S-curvature.

Lemma 3.1. ([5]) Let $F = \alpha\phi(\beta/\alpha)$ be an (α, β) -metric on an n -manifold. Then, F is of isotropic S-curvature $\mathbf{S} = (n+1)cF$, if and only if one of the following holds

(i) β satisfies

$$r_{ij} = \varepsilon \{b^2 a_{ij} - b_i b_j\}, \quad s_j = 0, \tag{8}$$

where $\varepsilon = \varepsilon(x)$ is a scalar function, and $\phi = \phi(s)$ satisfies

$$\Phi = -2(n+1)k \frac{\phi \Delta^2}{b^2 - s^2}, \tag{9}$$

where k is a constant. In this case, $c = k\varepsilon$.

(ii) β satisfies

$$r_{ij} = 0, \quad s_j = 0. \quad (10)$$

In this case, $c = 0$.

Let

$$\begin{aligned} \Psi_1 &:= \sqrt{b^2 - s^2} \Delta^{\frac{1}{2}} \left[\frac{\sqrt{b^2 - s^2} \Phi}{\Delta^{\frac{3}{2}}} \right]', \\ \Psi_2 &:= 2(n+1)(Q - sQ') + 3 \frac{\Phi}{\Delta}, \\ \theta &:= \frac{Q - sQ'}{2\Delta}. \end{aligned} \quad (11)$$

Then the formula for the mean Cartan torsion of an (α, β) -metric is given by following

$$\begin{aligned} I_i &= \frac{1}{2} \frac{\partial}{\partial y^i} \left[(n+1) \frac{\phi'}{\phi} - (n-2) \frac{s\phi''}{\phi - s\phi'} - \frac{3s\phi'' - (b^2 - s^2)\phi'''}{(\phi - s\phi') + (b^2 - s^2)\phi''} \right] \\ &= -\frac{\Phi(\phi - s\phi')}{2\Delta\phi\alpha^2} (\alpha b_i - sy_i). \end{aligned} \quad (12)$$

In [7], it is proved that the condition $\Phi = 0$ characterizes the Riemannian metrics among (α, β) -metrics. Hence, in the continue, we suppose that $\Phi \neq 0$.

Let $G^i = G^i(x, y)$ and $\bar{G}_\alpha^i = \bar{G}_\alpha^i(x, y)$ denote the coefficients of F and α respectively in the same coordinate system. By definition, we have

$$G^i = \bar{G}_\alpha^i + Py^i + Q^i, \quad (13)$$

where

$$\begin{aligned} P &:= \alpha^{-1} \Theta \left[-2Q\alpha s_0 + r_{00} \right] \\ Q^i &:= \alpha Q s_0^i + \Psi \left[-2Q\alpha s_0 + r_{00} \right] b^i. \end{aligned}$$

Simplifying (13) yields the following

$$G^i = \bar{G}_\alpha^i + \alpha Q s_0^i + \theta(-2\alpha Q s_0 + r_{00}) \left[\frac{y^i}{\alpha} + \frac{Q'}{Q - sQ'} b^i \right]. \quad (14)$$

Clearly, if β is parallel with respect to α ($r_{ij} = 0$ and $s_{ij} = 0$), then $P = 0$ and $Q^i = 0$. In this case, $G^i = \bar{G}_\alpha^i$ are quadratic in y , and F is a Berwald metric.

For an (α, β) -metric $F = \alpha\phi(s)$, the mean Landsberg curvature is given by

$$\begin{aligned} J_i &= -\frac{1}{2\Delta\alpha^4} \left[\frac{2\alpha^2}{b^2 - s^2} \left[\frac{\Phi}{\Delta} + (n+1)(Q - sQ') \right] (r_0 + s_0) h_i \right. \\ &\quad + \frac{\alpha}{b^2 - s^2} (\Psi_1 + s \frac{\Phi}{\Delta}) (r_{00} - 2\alpha Q s_0) h_i + \alpha \left[-\alpha Q' s_0 h_i + \alpha Q (\alpha^2 s_i - y_i s_0) \right. \\ &\quad \left. \left. + \alpha^2 \Delta s_{i0} + \alpha^2 (r_{i0} - 2\alpha Q s_i) - (r_{00} - 2\alpha Q s_0) y_i \right] \frac{\Phi}{\Delta} \right]. \end{aligned} \quad (15)$$

Contracting (15) with $b^i = a^{im}b_m$ yields

$$\bar{J} := J_i b^i = -\frac{1}{2\Delta\alpha^2} \left[\Psi_1(r_{00} - 2\alpha Q s_0) + \alpha \Psi_2(r_0 + s_0) \right]. \quad (16)$$

The horizontal covariant derivatives $J_{i;m}$ and $J_{i|m}$ of J_i with respect to F and α , respectively, are given by

$$\begin{aligned} J_{i;m} &= \frac{\partial J_i}{\partial x^m} - J_l \Gamma_{im}^l - \frac{\partial J_i}{\partial y^l} N_m^l, \\ J_{i|m} &= \frac{\partial J_i}{\partial x^m} - J_l \bar{\Gamma}_{im}^l - \frac{\partial J_i}{\partial y^l} \bar{N}_m^l. \end{aligned}$$

Then we have

$$J_{i;m} y^m = J_{i|m} y^m - J_l (N_i^l - \bar{N}_i^l) - 2 \frac{\partial J_i}{\partial y^l} (G^l - \bar{G}^l). \quad (17)$$

Let F be a Finsler metric of scalar flag curvature \mathbf{K} . By Akbar-Zadeh's theorem it satisfies following

$$A_{ijk;s;m} y^s y^m + \mathbf{K} F^2 A_{ijk} + \frac{F^2}{3} [h_{ij} \mathbf{K}_k + h_{jk} \mathbf{K}_j + h_{ki} \mathbf{K}_j] = 0, \quad (18)$$

where $A_{ijk} = FC_{ijk}$ is the Cartan torsion and $\mathbf{K}_i = \frac{\partial \mathbf{K}}{\partial y^i}$ [2]. Contracting (18) with g^{ij} yields

$$J_{i;m} y^m + \mathbf{K} F^2 I_i + \frac{n+1}{3} F^2 \mathbf{K}_i = 0. \quad (19)$$

By (17) and (19), for an (α, β) -metric $F = \alpha\phi(s)$ of constant flag curvature \mathbf{K} , the following holds

$$J_{i|m} - J_l \frac{\partial(G^l - \bar{G}^l)}{\partial y^i} b^i - 2 \frac{\partial \bar{J}}{\partial y^l} (G^l - \bar{G}^l) \mathbf{K} \alpha^2 \phi^2 I_i = 0. \quad (20)$$

Contracting (20) with b^i implies that

$$\bar{J}_{|m} y^m - J_i a^{ik} b_{k|m} y^m - J_l \frac{\partial(G^l - \bar{G}^l)}{\partial y^i} b^i - 2 \frac{\partial \bar{J}}{\partial y^l} (G^l - \bar{G}^l) + \mathbf{K} \alpha^2 \phi^2 I_i b^i = 0. \quad (21)$$

There exists a relation between mean Berwald curvature \mathbf{E} and the S-curvature \mathbf{S} . Indeed, taking twice vertical covariant derivatives of the S-curvature gives rise the E -curvature. It is easy to see that, every Finsler metric of isotropic S-curvature (2) is of isotropic mean Berwald curvature (5). Now, is the equation $\mathbf{S} = (n+1)cF$ equivalent to the equation $\mathbf{E} = \frac{n+1}{2}cF^{-1}\mathbf{h}$?

Recently, Cheng-Shen prove that a Randers metric $F = \alpha + \beta$ is of isotropic S-curvature if and only if it is of isotropic E -curvature [4]. Then, Chun-Huan-Cheng extend this equivalency to the Finsler metric $F = \alpha^{-m}(\alpha + \beta)^{m+1}$ for every real constant m , including Randers metric [3]. In [8], Cui extend their result and show that for the Matsumoto metric $F = \frac{\alpha^2}{\alpha - \beta}$ and the special (α, β) -metric $F = \alpha + \epsilon\beta + \kappa(\beta^2/\alpha)$ ($\kappa \neq 0$), these notions are equivalent.

To prove Theorem 1.1, we need the following.

Proposition 3.2. Let $F = \alpha + \beta + \frac{\beta^2}{\alpha} + \frac{\beta^3}{\alpha^2}$ be a second approximate Matsumoto metric on a manifold M of dimension n . Then the following are equivalent

- (i) F has isotropic S -curvature, $\mathbf{S} = (n+1)c(x)F$;
- (ii) F has isotropic mean Berwald curvature, $\mathbf{E} = \frac{n+1}{2}c(x)F^{-1}\mathbf{h}$;

where $c = c(x)$ is a scalar function on the manifold M . In this case, $\mathbf{S} = 0$. Then β is a Killing 1-form with constant length with respect to α , that is, $r_{00} = 0$.

Proof. (i) \rightarrow (ii) is obvious. Conversely, suppose that F has isotropic mean Berwald curvature, $\mathbf{E} = \frac{(n+1)}{2}c(x)F^{-1}\mathbf{h}$. Then we have

$$\mathbf{S} = (n+1)[cF + \eta], \quad (22)$$

where $\eta = \eta_i(x)y^i$ is a 1-form on M . For the second approximate Matsumoto metric, (6) reduces to following

$$\begin{aligned} Q &= -\frac{1+2s+3s^2}{-1+s^2+2s^3}, \\ \Theta &= \frac{1}{2} \frac{1-6s^2-12s^3-15s^4-12s^5}{(1+s+s^2+s^3)(1-3s^2-8s^3+2b^2+6b^2s)}, \\ \Psi &= \frac{1+3s}{(1-3s^2-8s^3+2b^2+6b^2s)}. \end{aligned} \quad (23)$$

By substituting (22) and (23) in (7), we have

$$\begin{aligned} \mathbf{S} &= \left[\frac{2(1+3s)(1+s+s^2+s^3)}{(-1+s^2+2s^3)^2} + \frac{2(1+3s)(1+2s+3s^2)s}{(1-3s^2-8s^3+2b^2+6b^2s)(-1+s^2+2s^3)} \right. \\ &\quad - \frac{2(5+26s+77s^2+88s^3-61s^4-430s^5-805s^6+4b^2+40b^2s+148b^2s^2)(b^2-s^2)}{(1-3s^2-8s^3+2b^2+6b^2s)^2(-1+s^2+2s^3)^2} \\ &\quad - \frac{2(256s^3b^2+252s^4b^2+216s^5b^2+108s^6b^2-828s^7-432s^8)(b^2-s^2)}{(1-3s^2-8s^3+2b^2+6b^2s)^2(-1+s^2+2s^3)^2} \\ &\quad \left. + \frac{(n+1)(1+2s+3s^2)(1-6s^2-12s^3-15s^4-12s^5)}{(-1+s^2+2s^3)(1+s+s^2+s^3)(1-3s^2-8s^3+2b^2+6b^2s)} + 2\lambda \right] s_0 \\ &\quad + 2 \left[\frac{(1+3s)}{1-3s^2-8s^3+2b^2+6b^2s} + \lambda \right] + \left[\frac{3(b^2-s^2)(1+11s^2+16s^3+2s)}{\alpha(1-3s^2-8s^3+2b^2+6b^2s)^2} \right] r_{00} \\ &\quad + \left[\frac{(n+1)(1-6s^2-12s^3-15s^4-12s^5)}{2\alpha(1+s+s^2+s^3)(1-3s^2-8s^3+2b^2+6b^2s)} \right] r_{00} \\ &= (n+1)[c\alpha(1+s+s^2+s^3) + \eta]. \end{aligned} \quad (24)$$

Multiplying (24) with $(-1+s^2+2s^3)(1+s+s^2+s^3)(1-3s^2-8s^3+2b^2+6b^2s)^2\alpha^{14}$ implies that

$$\begin{aligned} &M_1 + M_2\alpha^2 + M_3\alpha^4 + M_4\alpha^6 + M_5\alpha^8 + M_6\alpha^{10} + M_7\alpha^{12} + M_8\alpha^{14} \\ &\quad + \alpha \left[M_9 + M_{10}\alpha^2 + M_{11}\alpha^4 + M_{12}\alpha^6 + M_{13}\alpha^8 + M_{14}\alpha^{10} \right. \\ &\quad \left. + M_{15}\alpha^{12} + M_{16}\alpha^{14} \right] = 0, \end{aligned} \quad (25)$$

where

$$\begin{aligned}
M_1 &:= -128(n+1)c\beta^{15}, \\
M_2 &:= 2\left[(n+1)[(-385+96b^2)c\beta^2-64\eta\beta]+128\lambda(r_0+s_0)\beta+48nr_{00}\right]\beta^{11}, \\
M_3 &:= -\left[(n+1)[4(-283b^2+243+18b^4)c\beta^2-6(32b^2-59)\eta\beta\right. \\
&\quad + [12(-59+32b^2)\lambda(r_0+s_0)-96((3n-1)s_0-r_0)]\beta \\
&\quad \left.+ 3(-24b^2+36-219n+72nb^2)r_{00}\right]\beta^9, \\
M_4 &:= \left[(n+1)[-2(29-738b^2+208b^4)c\beta^2-3(-172b^2+21+24b^4)\eta\beta\right. \\
&\quad + (-159nb^2-132-39n+96b^2)r_{00}-3(-48b^2+123+72nb^2-277n)s_0\beta \\
&\quad \left.+ 3(24b^2+86)r_0\beta+6\lambda(-172b^2+21+24b^4)(s_0+r_0)\beta\right]\beta^7, \\
M_5 &:= \left[(n+1)[-8(-27b^2+70b^4-41)c\beta^2-4(47b^4-40-32b^2)\eta\beta\right. \\
&\quad + (15nb^2+20-55n+108b^2)r_{00}+8\lambda(47b^4-40-32b^2)(r_0+s_0)\beta \\
&\quad \left.- 2[(-262b^2+278+303nb^2-147n)s_0-(94b^2+32)r_0]\beta\right]\beta^5 \\
M_6 &:= 2\left[(n+1)[-2(4b+1)(4b-1)(4b^2+13)c\beta^2-10(1+20b^2+6b^4)\eta\beta\right. \\
&\quad + (11n-40b^2+35nb^2+20)r_{00}+20\lambda(1+20b^2+6b^4)(r_0+s_0)\beta \\
&\quad \left.- 2(32n+51+129nb^2-294b^2)s_0\beta-(30b^2-50)r_0\beta\right]\beta^3, \\
M_7 &:= -\left[(n+1)[-4(-17b^2-7+30b^4)c\beta^2-6(-1+10b^4)\beta]+(3n+12)b^2r_{00}\right. \\
&\quad \left.- 12\lambda(1-10b^4)(s_0+r_0)-6[(nb^2-n-2+14b^2)s_0-10b^2r_0]\beta\right]\beta, \\
M_8 &:= 4\left[(n+1)[2(1+2b^2)(8b^2+1)c\beta+(1+2b^2)^2\eta]-2\lambda(1+2b^2)^2(s_0+r_0)\right. \\
&\quad + [-57nb^2-64(n+1)b^4-8n-55b^2+6]r_{00}-(2-4b^2)r_0 \\
&\quad \left.+ [n+(4+2n)b^2-1]s_0\right], \\
M_9 &:= -416(n+1)c\beta^{14}, \\
M_{10} &:= \left[(n+1)[(-1037+616b^2)c\beta^2-288\eta\beta]+576\lambda(r_0+s_0)\beta+(204n-6)r_{00}\right]\beta^{12}, \\
M_{11} &:= -\frac{1}{2}\left[(n+1)[8(57b^4-385b^2+143)c\beta^2-2(424b^2-267)\eta\beta\right. \\
&\quad + [4\lambda(424b^2-267)(r_0+s_0)-4(-115+330n)s_0+1320r_0]\beta \\
&\quad \left.+ (300nb^2+249-189n-120b^2)r_{00}\right]\beta^8, \\
M_{12} &:= 4\left[(n+1)[-(572b^4-932b^2-275)c\beta^2-4(39b^4-28-102b^2)\eta\beta\right. \\
&\quad - 75nb^2-62-89n+144b^2r_{00}+8\lambda(39b^4-28-102b^2)(r_0+s_0)\beta \\
&\quad \left.- 6[(-58b^2+100+81nb^2-109n)s_0-(26b^2+34)r_0]\beta\right]\beta^6,
\end{aligned}$$

$$\begin{aligned}
M_{13} &:= \left[(n+1)[-8(-24+35b^2+47b^4)c\beta^2 - 6(26b^4-11+20b^2)\eta\beta] \right. \\
&\quad + (51nb^2+33-6n+24b^2)r_{00} + 12\lambda(26b^4-11+20b^2)(s_0+r_0) \\
&\quad \left. - 6(-118b^2+54+83nb^2-3n)s_0\beta + 6(26b^2-10)r_0\beta \right] \beta^4, \\
M_{14} &:= - \left[(n+1)[-(56b^4-272b^2-39)c\beta^2 - 4(7(n+1)b^4-6(1+n)-22nb^2)\eta\beta] \right. \\
&\quad + (-7nb^2-8-5n+32b^2)r_{00} + 8\lambda(7b^4-6-22b^2)(r_0+s_0)\beta \\
&\quad \left. + 2(-154b^2+12+33nb^2+19n)s_0\beta + 2(14b^2-22)s_0\beta \right] \beta^2 \\
M_{15} &:= \left[(n+1)[4(23b^4+5b^2-1)c\beta^2 - (14b^2+1)(2b^2+1)\eta\beta] \right. \\
&\quad - \frac{n+1}{2}(8b^2+1)r_{00} - 2\lambda(14b^2+1)(2b^2+1)(s_0+r_0)\beta \\
&\quad \left. - 2(-6b^2+3-5nb^2)s_0\beta - 2(4+14b^2)r_0\beta \right], \\
M_{16} &:= (n+1)(1+2b^2)^2c
\end{aligned}$$

The term of (25) which is seemingly does not contain α^2 is M_1 . Since β^{15} is not divisible by α^2 , then $c = 0$ which implies that

$$M_1 = M_9 = 0.$$

Therefore (25) reduces to following

$$M_2 + M_3\alpha^2 + M_4\alpha^4 + M_5\alpha^6 + M_6\alpha^8 + M_7\alpha^{10} + M_8\alpha^{12} = 0, \quad (26)$$

$$M_{10} + M_{11}\alpha^2 + M_{12}\alpha^4 + M_{13}\alpha^6 + M_{14}\alpha^8 + M_{15}\alpha^{10} + M_{16}\alpha^{12} = 0. \quad (27)$$

By plugging $c = 0$ in M_2 and M_{10} , the only equations that don't contain α^2 are the following

$$8 \left[8(2\lambda(r_0+s_0) - (n+1)\eta) + 6nr_{00} \right] = \tau_1\alpha^2, \quad (28)$$

$$6 \left[48(2\lambda(r_0+s_0) - (n+1)\eta) + (34n-1)r_{00} \right] = \tau_2\alpha^2, \quad (29)$$

where $\tau_1 = \tau_1(x)$ and $\tau_2 = \tau_2(x)$ are scalar functions on M . By eliminating $[2\lambda(r_0+s_0) - (n+1)\eta]$ from (28) and (29), we get

$$r_{00} = \tau\alpha^2, \quad (30)$$

where $\tau = \frac{\tau_2 - \tau_1}{-(18n+1)}$. By (28) or (29), it follows that

$$2\lambda(r_0+s_0) - (n+1)\eta = 0. \quad (31)$$

By (30), we have $r_0 = \tau\beta$. Putting (30) and (31) in M_{10} and M_{11} yield

$$M_{10} = (204n-6)\tau\alpha^2\beta^{12}, \quad (32)$$

$$M_{11} = \left[[(660n-230)s_0 - 660r_0]\beta - \frac{(300n-120)b^2 + 249 - 189n}{2}r_{00}\tau\alpha^2 \right] \beta^3 \quad (33)$$

By putting (32) and (33) into (27), we have

$$[(660n - 230)s_0 - 660r_0]\beta^{10} - \frac{300nb^2 + 249 - 189n - 120b^2}{2}r_{00}\tau\alpha^2\beta^9 + (204n - 6)\tau\beta^{12} - M_{12}\alpha^2 + M_{13}\alpha^4 + M_{14}\alpha^6 + M_{15}\alpha^8 + M_{16}\alpha^{10} = 0. \quad (34)$$

The only equations of (34) that do not contain α^2 is $[(204n - 6)\tau\beta^2 + (660n - 230)s_0 - 660r_0]\beta^{10}$. Since β^{10} is not divisible by α^2 , then we have

$$[(204n - 6)\tau\beta^2 + (660n - 230)s_0 - 660r_0] = 0. \quad (35)$$

By Lemma 3.1, we always have $s_j = 0$. Then (35), reduces to following

$$(204n - 6)\tau\beta^2 - 660r_0 = 0. \quad (36)$$

Thus

$$2(204n - 6)\tau b_i \beta - 660\tau b_i = 0. \quad (37)$$

By multiplying (37) with b^i , we have

$$\tau = 0.$$

Thus by (31), we get $\eta = 0$ and then $\mathbf{S} = (n + 1)cF$. By (30), we get $r_{ij} = 0$. Therefore Lemma 3.1, implies that $\mathbf{S} = 0$. This completes the proof. \square

Proof of Theorem 1.1: Let F be an isotropic Berwald metric (3) with almost isotropic flag curvature (1). In [20], it is proved that every isotropic Berwald metric (3) has isotropic S-curvature (2).

Conversely, suppose that F is of isotropic S-curvature (2) with scalar flag curvature \mathbf{K} . In [11], it is showed that every Finsler metric of isotropic S-curvature (2) has almost isotropic flag curvature (1). Now, we are going to prove that F is a isotropic Berwald metric. In [6], it is proved that F is an isotropic Berwald metric (3) if and only if it is a Douglas metric with isotropic mean Berwald curvature (5). On the other hand, every Finsler metric of isotropic S-curvature (2) has isotropic mean Berwald curvature (5). Thus for completing the proof, we must show that F is a Douglas metric. By Proposition 3.2, we have $\mathbf{S} = 0$. Therefore by Theorem 1.1 in [11], F must be of isotropic flag curvature $\mathbf{K} = \sigma(x)$. By Proposition 3.2, β is a Killing 1-form with constant length with respect to α , that is, $r_{ij} = s_j = 0$. Then (14), (15) and (16) reduce to

$$G^i - \bar{G}^i = \alpha Q s^i_{0}, \quad J_i = -\frac{\Phi s_{i0}}{2\alpha\Delta}, \quad \bar{J} = 0. \quad (38)$$

By (12), we get

$$I_i b^i = \frac{-\Phi}{2\Delta F}(\phi - s\phi')(b^2 - s^2). \quad (39)$$

We consider two case:

Case 1. Let $\dim M \geq 3$. In this case, by Schur Lemma F has constant flag curvature and (21) holds. Thus by (38) and (39), the equation (21) reduces to following

$$\frac{\Phi s_{i0}}{2\alpha\Delta} a^{ik} s_{k0} + \frac{\Phi s_{l0}}{2\alpha\Delta} \left(\alpha Q s^l_0 \right)_{.i} b^i - \mathbf{K} F \frac{\Phi}{2\Delta} (\phi - s\phi')(b^2 - s^2) = 0. \quad (40)$$

By assumption $\Phi \neq 0$. Thus by (40), we get

$$s_{i0} s^i_0 + s_{l0} \left(\alpha Q s^l_0 \right)_{.i} b^i - \mathbf{K} F \alpha (\phi - s\phi')(b^2 - s^2) = 0. \quad (41)$$

The following holds

$$\left(\alpha Q s^l_0 \right)_{.i} b^i = s Q s^i_0 + Q' s^i_0 (b^2 - s^2).$$

Then (41) can be rewritten as follows

$$s_{i0} s^i_0 \Delta - \mathbf{K} \alpha^2 \phi (\phi - s\phi')(b^2 - s^2) = 0. \quad (42)$$

By (11), (23) and (42), we obtain

$$\left[1 - \frac{s(1+2s+3s^2)}{(-1+s^2+2s^3)} + \frac{2(b^2-s^2)(1+3s)(1+s+s^2+s^3)}{(-1+s^2+2s^3)^2} \right] s_{i0} s^i_0 - \mathbf{K} (1+s+s^2+s^3) \alpha^2 [1+s+s^2+s^3-s(1+2s+3s^2)] (b^2-s^2) = 0 \quad (43)$$

Multiplying (43) with $(-1+s^2+2s^3)^2 \alpha^{12}$ yields

$$A + \alpha B = 0,$$

where

$$\begin{aligned} A &= -\mathbf{K} b^2 \alpha^{14} + (2b^2 + 1)(\mathbf{K} \beta^2 + s_{i0} s^i_0) \alpha^{12} \\ &\quad + 2(3\mathbf{K} b^2 \beta^2 + 4s_{i0} s^i_0 b^2 - \mathbf{K} \beta^2 - s_{i0} s^i_0) \beta^2 \alpha^{10} \\ &\quad - (6\mathbf{K} \beta^2 + 11s_{i0} s^i_0 + 20\mathbf{K} \beta^2 b^2 - 6s_{i0} s^i_0) \beta^4 \alpha^8 \\ &\quad - (-20\mathbf{K} \beta^2 + 5\mathbf{K} \beta^2 b^2 + 8s_{i0} s^i_0) \beta^6 \alpha^6 + (\mathbf{K} \beta^{10})(26b^2 + 5) \alpha^4 \\ &\quad - 2\mathbf{K} \beta^{12} (13 - 4b^2) \alpha^2 - 8\mathbf{K} \beta^{14}, \\ B &= -(\mathbf{K} b^2 \beta) \alpha^{12} + (1 + 8b^2)(\mathbf{K} b^2 \beta^2 + s_{i0} s^i_0) \beta \alpha^{10} \\ &\quad - 2(3\mathbf{K} b^2 \beta^2 - 4s_{i0} s^i_0 b^2 + 4\mathbf{K} \beta^2 + 5s_{i0} s^i_0) \beta^3 \alpha^8 \\ &\quad + (6\mathbf{K} \beta^2 - 11s_{i0} s^i_0 - 20\mathbf{K} \beta^2 b^2) \beta^5 \alpha^6 \\ &\quad + (5\mathbf{K} \beta^9)(3b^2 + 4) \alpha^4 + (5\mathbf{K} \beta^{11})(-3 + 4b^2) \alpha^2 - 20\alpha \mathbf{K} \beta^{13}. \end{aligned} \quad (44)$$

Obviously, we have $A = 0$ and $B = 0$.

By $A = 0$ and the fact that β^{14} is not divisible by α^2 , we get $\mathbf{K} = 0$. Therefore (43) reduces to following

$$s_{i0} s^i_0 = a_{ij} s^j_0 s^i_0 = 0.$$

Because of positive-definiteness of the Riemannian metric α , we have $s^i_0 = 0$, i.e., β is closed. By $r_{00} = 0$ and $s_0 = 0$, it follows that β is parallel with respect to α . Then $F = \alpha + \beta + \frac{\beta^2}{\alpha} + \frac{\beta^3}{\alpha^2}$ is a Berwald metric. Hence F must be locally Minkowskian.

Case 2. Let $\dim M = 2$. Suppose that F has isotropic Berwald curvature (3). In [20], it is proved that every isotropic Berwald metric (3) has isotropic S -curvature, $\mathbf{S} = (n+1)cF$. By Proposition 3.2, $c = 0$. Then by (3), F reduces to a Berwald metric. Since F is non-Riemannian, then by Szabó's rigidity Theorem for Berwald surface (see [2] page 278), F must be locally Minkowskian. \square

Acknowledgment. The authors would like to thank the referee's valuable suggestions.

References

- [1] S. Bácsó and M. Matsumoto, *On Finsler spaces of Douglas type, A generalization of notion of Berwald space*, Publ. Math. Debrecen. **51**(1997), 385-406.
- [2] D. Bao, S. S. Chern and Z. Shen, *An Introduction to Riemann-Finsler Geometry*, Springer-Verlag, 2000.
- [3] X. Chun-Huan and X. Cheng, *On a class of weakly-Berwald (α, β) -metrics*, J. Math. Res. Expos. **29**(2009), 227-236.
- [4] X. Cheng and Z. Shen, *Randers metric with special curvature properties*, Osaka. J. Math. **40**(2003), 87-101.
- [5] X. Cheng and Z. Shen, *A class of Finsler metrics with isotropic S -curvature*, Israel. J. Math. **169**(2009), 317-340.
- [6] X. Chen and Z. Shen, *On Douglas metrics*, Publ. Math. Debrecen. **66**(2005), 503-512.
- [7] X. Cheng, H. Wang and M. Wang, *(α, β) -metrics with relatively isotropic mean Landsberg curvature*, Publ. Math. Debrecen. **72**(2008), 475-485.
- [8] N. Cui, *On the S -curvature of some (α, β) -metrics*, Acta. Math. Scientia, Series: A. **26**(7) (2006), 1047-1056.
- [9] I.Y. Lee and M.H. Lee, *On weakly-Berwald spaces of special (α, β) -metrics*, Bull. Korean Math. Soc. **43**(2) (2006), 425-441.
- [10] M. Matsumoto, *Theory of Finsler spaces with (α, β) -metric*, Rep. Math. Phys. **31**(1992), 43-84.

- [11] B. Najafi, Z. Shen and A. Tayebi , *Finsler metrics of scalar flag curvature with special non-Riemannian curvature properties*, Geom. Dedicata. **131**(2008), 87-97.
- [12] H. S. Park and E. S. Choi, *On a Finsler spaces with a special (α, β) -metric*, Tensor, N. S. **56**(1995), 142-148.
- [13] H. S. Park and E. S. Choi, *Finsler spaces with an approximate Matsumoto metric of Douglas type*, Comm. Korean. Math. Soc. **14**(1999), 535-544.
- [14] H. S. Park and E. S. Choi, *Finsler spaces with the second approximate Matsumoto metric*, Bull. Korean. Math. Soc. **39**(1) (2002), 153-163.
- [15] H. S. Park, I. Y. Lee and C. K. Park, *Finsler space with the general approximate Matsumoto metric*, Indian J. Pure. Appl. Math. **34**(1) (2002), 59-77.
- [16] H. S. Park, I.Y. Lee, H. Y. Park and B. D. Kim, *Projectively flat Finsler space with an approximate Matsumoto metric*, Comm. Korean. Math. Soc. **18**(2003), 501-513.
- [17] Z. Shen, *Differential Geometry of Spray and Finsler Spaces*, Kluwer Academic Publishers, Dordrecht, 2001.
- [18] A. Tayebi and B. Najafi, *On isotropic Berwald metrics*, Ann. Polon. Math. **103**(2012), 109-121.
- [19] A. Tayebi, E. Peyghan and H. Sadeghi, *On Matsumoto-type Finsler metrics*, Nonlinear Analysis: RWA. **13**(2012), 2556-2561.
- [20] A. Tayebi and M. Rafie Rad, *S-curvature of isotropic Berwald metrics*, Sci. China. Series A: Math. **51**(2008), 2198-2204.

Akbar Tayebi and Tayebeh Tabatabaeifar
 Faculty of Science, Department of Mathematics
 University of Qom
 Qom. Iran
 Email: akbar.tayebi@gmail.com
 Email: t.tabaei@gmail.com

Esmail Peyghan
 Faculty of Science, Department of Mathematics
 Arak University 38156-8-8349
 Arak. Iran
 Email: epeyghan@gmail.com